**Lecture Sheet**

**On**

**Inequality**

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**Inequality**

**Inequality**: An inequality is a statement which expresses a non-equal relationship between two mathematical expressions.

The followings indicate the meaning of inequality signs:

1.  means is greater than.
2.  means is less than.
3.  means is greater than or equal to .
4.  means is less than or equal to .

**Some rules of inequality:** For all real numbers a, b and c:

1. If then or or .
2. If  then .
3. If  or then .
4. If  and or and  then .
5. If  and  then  , , and .
6. If  and  then  and .
7. If  then ,  and  where .

**Problem-01:** If  and , then show that .

**Solution**: Since  and , so the relation between *a* and *b* must be  or .

Now 





Here, it follows that if  or then .

 **(Showed).**

**Problem-02:** If  and , then show that .

**Solution**: Here , 





Here, it follows that if  and then .



 **(Showed).**

**Problem-03:** Suppose a, b, c, d are positive real numbers. If , then show that .

**Solution**: Here , 









 **(Showed).**

**Problem-04:** If  and , then show that.

**Solution**: If  and , then .

Now adding these we get,



Again since  and , so



Multiplying (1) and (2), we get



 **(Showed).**

**Problem-05:** If  and , then show that .

**Solution**: Here, 









Since  so we get,



Now the sum of these inequalities gives us,





From (1) and (2), we get





 **(Showed).**

**Problem-06:** If , then show that .

**Solution**: We know, 

Then 



Again, we can write,















From (1) and (2), we get

 **(Showed).**

**Arithmetic and Geometric Means:** Let  represent  positive numbers. The arithmetic and geometric mean of these numbers are defined as follows,



.

**Theorem-01:** Prove that the arithmetic mean (A) of  positive numbers is greater than or equal to their geometric mean (G).

**Proof:** Let  are  positive numbers. The arithmetic and geometric mean of these numbers are defined as follows,



.

Case-01: Suppose the numbers are not all equal to one another. Then we know,





Similarly, .

Therefore, .

Proceeding in this way , we can show that if  is a power of 2, then









If  is not a power of 2, consider the set , where *A* occurs  times and  is a power of 2. By the preceding,















This implies that the arithmetic mean of the numbers is greater than their geometric mean.

Case-02: Suppose the numbers are all equal to one another i.e. . Then we have,



.

This implies that the arithmetic mean of the numbers is equal to their geometric mean.

Hence, it is concluded that the arithmetic mean (A) of  positive numbers is greater than or equal to their geometric mean (G). (**Proved**)

**Problem-07:** Show that .

**Solution**: We know, 





 **(Showed).**

**Problem-08:** Show that .

**Solution**: We know, 

So





Again, 



Again, 



Adding (1), (2) and (3), we get



 **(Showed).**

**Exercise:**

**Problem-01:** If, then show that

1. .
2. .
3. .
4. .
5. .
6. .

**Problem-09:** If and , then show that .

**Answer:** We know, 

Here, 





Again we know,









Now from (1) and (2), we get



 **(Showed).**

**Theorem-02:** If and , then show that  when  .

**Answer:** We know,



Since , so expanding (1) as a power series of  and then dividing by 2 , we get



Case-01: If  but , or if  is negative number, then the right hand side of (2) is always positive.

So in this case,

.

Case-02: If  , then in the right hand side of (2) all terms are negative except first term.

So in this case,

.

Case-03: If  and let  where , then





. **(Showed).**

**Problem-10:** If and all are not equal to one another, then show that

1.  when .
2.  when .

**Answer:** We know, if  and , then



and 

1. Suppose, 



Putting  and  consecutively in (1) and then multiplying each time by  , we get









Now adding these, we get









. **(Showed).**

1. Similarly from (2) we can show that

. **(Showed).**

**Probem-11:** If, then show that  .

**Answer:** We know,





. **(Showed).**

**Theorem-03:** State and Prove Weiestras’s Inequality.

**Answer:** **Statement:** If  and , then

1. .
2.  where .
3.  where .
4. .

**Proof:** (a). Here, 



Since , so we can write,

. **(Proved).**

(b) Here, .

 where .

Similarly, 

 where .

Proceeding in the same way, we can write

 where 

  .

(c) Here, 

.

Similarly, .

.



.

Now multiplying these, we get



If , then using (2) in (3), we have

 **(Proved).**

(d) Here, 

.

Similarly, .

.



.

Now multiplying these, we get



Using (1) in (4), we have

 **(Proved).**

**Theorem-04:** State and Prove Cauchy-Schwarz’s Inequality.

**Answer:** **Statement:** If  and  are two sets of real numbers, then



the sign of equality occurring only when .

**Proof:** For all real numbers,, we can write,







where , , and .

Dividing both sides of (1) by *A*, we have







For all value of , (2) is true if











Suppose the sets are proportional i.e.

.

Now using (4) in (3), we get









. **(Proved).**